On the cycle map for products of elliptic curves over a p-adic field

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Abstract

We study the Chow group of 0-cycles on the product of elliptic curves over a p-adic field. For this abelian variety, it is decided that the structure of the image of the Albanese kernel by the cycle class map.

1 Introduction

Let $X = E \times E'$ be the product of elliptic curves E and E' defined over a finite extension K of the p-adic field $\mathbb{Q}_p$. The main objective of this note is to study the Chow group $\mathrm{CH}_0(X)$ of 0-cycles on X modulo rational equivalence. Let $A_0(X)$ be the kernel of the degree map $\mathrm{CH}_0(X) \to \mathbb{Z}$ and $T(X)$ the kernel of the Albanese map $A_0(X) \to X(K)$ so called the Albanese kernel for X. These maps are surjective, and we have $\mathrm{CH}_0(X)/T(X) \simeq \mathbb{Z} \oplus X(K)$. If we assume p^n-torsion points $E[p^n]$ and $E'[p^n]$ are K-rational, Mattuck's theorem [11] on $X(K)$ implies $\mathrm{CH}_0(X)/p^n \simeq (\mathbb{Z}/p^n)^{\oplus(2[K:\mathbb{Q}_p]+5)} \oplus T(X)/p^n$. Raskind-Spieß [16] showed the injectivity of the cycle map $\rho : T(X)/p^n \to H^4(X, \mathbb{Z}/p^n(2))$ to the étale cohomology group of X with coefficients $\mathbb{Z}/p^n(2) = \mu_{p^n} \otimes \mu_{p^n}$ when E and E' have ordinary or split multiplicative reduction. Although it is difficult to know the kernel of ρ in general (the injectivity fails for certain surfaces, see [15], Sect. 8), one can calculate the structure of its image. This is the main theorem of this note:

Theorem (Thm. 3.4)**.** *Let E and E' be elliptic curves over K with good or split multiplicative reduction, and $E[p^n]$ and $E'[p^n]$ are K-rational. The structure of the image of $T(X)/p^n$ for $X = E \times E'$ by the cycle map ρ is*

(i) $\mathbb{Z}/p^n$ *if both* E *and* E' *have ordinary or split multiplicative reduction.*

(ii) $\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$ *if* E *and* E' *have different reduction types.*

The same computation works well in the remained case: Both of E and E' have supersingular reduction. The image may be varied according to the p-th coefficients of multiplication p formula of the formal completion of the elliptic curves along the origin (*cf.* Prop. 3.6). For an arbitrary elliptic curves E, E' over K and $X = E \times E'$, the base change $X' := X \otimes_K K'$ to some sufficiently large extension field K' over K satisfies the assumptions in our main theorem above. Since the kernel of the multiplication by p^n on $\mathrm{CH}_0(X)$ is finite (due to Colliot-Thélène, [4]), we have a surjection $\mathrm{CH}_0(X')/p^n \to \mathrm{CH}_0(X)/p^n$ with finite kernel if we admit Raskind and Spieß's conjecture ([16], Conj. 3.5.4); the finiteness of the kernel of the cycle map on X' (*cf.* [16], Cor. 3.5.2). Therefore, we limit our consideration as in the above theorem. The estimation of the difference of the image of $T(X)/p^n$ and $T(X')/p^n$ by the cycle maps is also a problem. Murre and Ramakrishnan ([12], Thm. A) gave an answer to this problem in the case of $n = 1$ for the self-product $X = E \times E$ of an elliptic curve E over K with ordinary good reduction. In this case, they proved that the structure of the image is at most $\mathbb{Z}/p$ and is exactly $\mathbb{Z}/p$ if and only if the definition field $K(E[p])$ over K is unramified with the prime to p-part of $[K(E[p]) : K]$ is ≤ 2 and K has a p-th root of unity ζ_p.

The results in our main theorem are known by Takemoto [20] in the case of ordinary reduction or split multiplicative reduction. So our main interest is in supersingular elliptic curves. In Section 2 we study the image of the Kummer homomorphism associated with isogeny of formal groups. The main ingredient is the structure of the graded quotients of a filtration on the formal groups (Prop. 2.8). As a special case, we obtain the structure of the graded quotients associated with filtration on the multiplicative group modulo p^n. In Appendix, we show that the results work also on the Milnor K-groups more generally. The proof of the main theorem is given in Section 3.

For a discrete valuation field K, we denote by $\mathcal{O}_K$ the valuation ring of K, $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, $k := \mathcal{O}_K/\mathfrak{m}_K$ the residue field of $\mathcal{O}_K$, v_K the normalized valuation of $\mathcal{O}_K$, $\mathcal{O}_K^\times$ the group of units in $\mathcal{O}_K$, $\overline{K}$ a fixed separable closure of K and $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. For an abelian group A and a non-zero integer m, let $A[m]$ be the kernel and A/m the cokernel of the map $m : A \to A$ defined by multiplication by m.

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2 Formal Groups

Let K be a complete discrete valuation field of characteristic 0, and k its perfect residue field of characteristic $p > 0$. In this section, we decide the image of the Kummer map associated with an isogeny of formal groups (Thm. 2.11). First we recall some basic notions on formal groups from [6]. Throughout this section, all formal groups are commutative of dimension one. Let F be a formal group over the valuation ring $\mathcal{O}_K$. The elements of the maximal ideal $\mathfrak{m}_{\overline{K}}$ of $\mathcal{O}_{\overline{K}}$ form a G_K-module denoted by $F(\overline{K})$ under the operation $x + y := F(x, y)$. Similarly, for a finite extension L/K, the maximal ideal $\mathfrak{m}_L$ forms a subgroup of $F(\overline{K})$ denoted by $F(L)$. For an isogeny $\phi : F \to G$ of formal groups defined over $\mathcal{O}_K$, we regard it as a power series $\phi(T) = a_1T + a_2T^2 + \cdots + a_pT^p + \cdots \in \mathcal{O}_K[\![T]\!]$. The coefficient of T in $\phi(T)$ is denoted by $D(\phi) := a_1$. The *height* of ϕ is defined to be a positive integer n such that $\phi(T) \equiv \psi(T^{p^n}) \mod \mathfrak{m}_K$ for some $\psi \in \mathcal{O}_K[\![T]\!]$ with $v_K(D(\psi)) = 0$ (*cf.* [9], 2.1). It is known that the induced homomorphism $F(\overline{K}) \to G(\overline{K})$ from the isogeny $\phi : F \to G$ is surjective and the kernel of ϕ (= the kernel of the homomorphism $F(\overline{K}) \to G(\overline{K})$ induced by ϕ) is a finite group of order p^n, where n is the height of ϕ. For any integer $m \geq 1$, $F^m(K)$ is the subgroup of $F(K)$ consisting of the set $\mathfrak{m}_K^m$. Fix a uniformizer π of K. For any $m \geq 1$, we have an isomorphism

$$\rho : k \xrightarrow{\simeq} \mathrm{gr}^m(F) := F^m(K)/F^{m+1}(K) \tag{1}$$

defined by $x \mapsto \widetilde{x}\pi^m$, where $\widetilde{x} \in \mathcal{O}_K^\times$ is a lift of $x \in k \smallsetminus \{0\}$. Recall the behavior of the operation on the graded quotients of raising to an isogeny $\phi : F \to G$ with height 1.

Lemma 2.1 ([1]; [9], Lem. 2.1.2)**.** *Let $\phi(T) := a_1T + a_2T^2 + \cdots$ be an isogeny $F \to G$ of formal groups defined over $\mathcal{O}_K$ with height* 1*.*

(i) *The coefficient a_p is a unit in $\mathcal{O}_K$.*

(ii) *For m such that $p \nmid m$, we have $a_1 \mid a_m$.*

The following lemma is proved essentially as same as the case $F = \widehat{\mathbb{G}}_m$ the multiplicative group (e.g., [5], Chap. I, Sect. 5).

Lemma 2.2. *Let $\phi(T) := a_1T + a_2T^2 + \cdots$ be an isogeny $F \to G$ of formal groups defined over $\mathcal{O}_K$ with height 1. Define $t := v_K(a_1)$ and let a be the residue class of $a_1\pi^{-t}$ and $m \geq 1$ an integer. Then, we have $\phi(F^m(K)) \subset G^{mp}(K)$ for $m \leq t/(p-1)$ and $\phi(F^m(K)) \subset G^{m+t}(K)$ for $m > t/(p-1)$. The isogeny ϕ induces the following:*
(i) *If $m < t/(p-1)$, the diagram*

$$\begin{tikzcd} \mathrm{gr}^m(F) \arrow[r, "\phi"] & \mathrm{gr}^{mp}(G) \\ k \arrow[u, "\rho"] \arrow[r, "\overline{a}_p C^{-1}"] & k \arrow[u, "\rho"'] \end{tikzcd}$$

is commutative, where $\overline{a}_p \in k$ is the residue class of $a_p \in \mathcal{O}_K^\times$ and $C^{-1} : k \to k$ is "the inverse Cartier operator" [1] *defined by $x \mapsto x^p$. The horizontal homomorphisms are bijective.*
(ii) *If $m = t/(p-1)$ is in $\mathbb{Z}$, the diagram*

$$\begin{tikzcd} \mathrm{gr}^{t/(p-1)}(F) \arrow[r, "\phi"] & \mathrm{gr}^{t+t/(p-1)}(G) \\ k \arrow[u, "\rho"] \arrow[r, "a+\overline{a}_p C^{-1}"] & k \arrow[u, "\rho"'] \end{tikzcd}$$

is commutative, where the bottom map defined by $x \mapsto ax + \overline{a}_p x^p$.
(iii) *If $m > t/(p-1)$, the diagram*

$$\begin{tikzcd} \mathrm{gr}^m(F) \arrow[r, "\phi"] & \mathrm{gr}^{m+t}(G) \\ k \arrow[u, "\rho"] \arrow[r, "a"] & k \arrow[u, "\rho"'] \end{tikzcd}$$

is commutative, where the bottom map defined by $x \mapsto ax$. The horizontal homomorphisms are bijective. Furthermore, we have $G^{m+t}(K) \subset \phi F^m(K)$.

Proof. Take any $u\pi^m \in F^m(K)$ with $u \in \mathcal{O}_K^\times$. From Lemma 2.1, we have $v_K(\phi(u\pi^m)) \geq \min\{t+m, pm\}$ (the equality holds if $m \neq t/(p-1)$). Moreover, we have

$$\phi(u\pi^m) \equiv \begin{cases} \overline{a}_p u^p \pi^{mp} \mod \pi^{mp+1}, & \text{if } m < t/(p-1), \\ (au + \overline{a}_p u^p)\pi^{t+t/(p-1)} \mod \pi^{t+t/(p-1)+1}, & \text{if } m = t/(p-1), \\ au\pi^{m+t} \mod \pi^{m+t+1}, & \text{if } m > t/(p-1). \end{cases}$$

[1] For the original definition of the inverse Cartier operator, see (11) in Appendix.

The assertions except the last one follow from it. Using the completeness of K, we obtain $G^{m+t}(K) \subset G^{m+t+1}(K) + \phi F^m(K) \subset G^{m+t+2}(K) + \phi F^m(K) \subset \cdots \subset \phi F^m(K)$ if $m > t/(p-1)$. $\square$

Corollary 2.3. *Let $\phi : F \to G$ be an isogeny of formal groups defined over $\mathcal{O}_K$ with height 1. Assume $F[\phi] := \mathrm{Ker}(\phi) \subset F(K)$. For any non-zero element $x \in F[\phi]$, we have $v_K(x) = t/(p-1) \in \mathbb{Z}$. The kernel of $\phi : \mathrm{gr}^{t/(p-1)}(F) \to \mathrm{gr}^{t+t/(p-1)}(G)$ is of order p.*

Proof. For any non-zero $x \in F[\phi]$, we have $\phi(x) = a_1 x + a_2 x^2 + \cdots = 0$. Hence $t + v_K(x) = v_K(a_1 x) = v_K(a_p x^p) = p v_K(x)$ and $v_K(x) = t/(p-1)$. The kernel of $x \mapsto ax + \overline{a}_p x^p$ is $\sqrt[p-1]{-a/\overline{a}_p}\mathbb{F}_p$. $\square$

The filtration $G^m(\phi)$ on $G(\phi) := G(K)/\phi F(K)$ is defined by the image of the filtration $G^m(K)$. For an isogeny $\phi : F \to G$ with height 1, its graded quotients $\mathrm{gr}^m(\phi) := G^m(\phi)/G^{m+1}(\phi)$ describe the cokernels of ϕ in Lemma 2.2 as follows:

Lemma 2.4. *Let $\phi : F \to G$ be an isogeny over $\mathcal{O}_K$ with height 1 and $t := v_K(D(\phi))$.*
(i) *If $m < t + t/(p-1)$, the following sequence*

$$0 \to \mathrm{gr}^{m/p}(F) \xrightarrow{\phi} \mathrm{gr}^m(G) \to \mathrm{gr}^m(\phi) \to 0$$

is exact, where $\mathrm{gr}^x(F) = 0$ if $x \notin \mathbb{Z}$ by convention.
(ii) *If $m = t + t/(p-1)$ is in $\mathbb{Z}$, then*

$$\mathrm{gr}^{t/(p-1)}(F) \xrightarrow{\phi} \mathrm{gr}^{t+t/(p-1)}(G) \to \mathrm{gr}^{t+t/(p-1)}(\phi) \to 0$$

is exact.
(iii) *If $m > t + t/(p-1)$, then*

$$0 \to \mathrm{gr}^{m-t}(F) \xrightarrow{\phi} \mathrm{gr}^m(G) \to \mathrm{gr}^m(\phi) \to 0$$

is exact.

Proof. Note that we have $\mathrm{gr}^m(\phi) \simeq G^m(K)/(\phi F(K) \cap G^m(K) + G^{m+1}(K))$. Consider the case (i), (ii). For any $\phi(x) \in G^m(K)$ with $x \in F(K)$, we have an inequality $v_K(\phi(x)) \geq \min\{t + r, pr\}$, where $r = v_K(x)$ (the equality holds if $r \neq t/(p-1)$) by Lemma 2.1. To show the injectivity of $\mathrm{gr}^m(G) \to$

$\mathrm{gr}^m(\phi)$ if $p \nmid m$, it is enough to show $\phi F(K) \cap G^m(K) \subset G^{m+1}(K)$. For any $\phi(x) \in G^m(K) \cap \phi F(K)$, assume $m = v_K(\phi(x))$. By Lemma 2.1 as above, $m = pr$ if $t/(p-1) > r$. Otherwise, $m = pt/(p-1)$. This contradicts to $p \nmid m$. Thus $v_K(\phi(x)) > m$ and we obtain $\phi(x) \in G^{m+1}(K)$. In the case of $p \mid m$, Take any $\phi(x) \in G^m(K) \cap \phi F(K)$ with $m = v_K(\phi(x))$. From the above (in)equality, we have $v_K(x) = m/p$. The rest of the assertions follows from it. Next we consider the case (iii). For any $\phi(x) \in G^m(K) \cap \phi F(K)$ with $m = v_K(\phi(x))$. If $t/(p-1) > r$ then $m = pr < pt/(p-1)$ and this contradicts to $m > pt/(p-1)$. Otherwise $m \geq t + r$. Hence $r \leq m - t$ and thus $x \in F^{m-t}(K)$. $\square$

Recall that a perfect field is said to be *quasi-finite* if its absolute Galois group is isomorphic to $\widehat{\mathbb{Z}}$ (*cf.* [17], Chap. XIII, Sect. 2).

Corollary 2.5 (*cf.* [1], Lem. 1.1.2; [9], Lem. 2.1.3)**.** *Let $\phi(T) := a_1 T + a_2 T^2 + \cdots$ be an isogeny $F \to G$ of formal groups defined over $\mathcal{O}_K$ with height* 1. *Assume $F[\phi] \subset F(K)$. Define $t := v_K(a_1)$ and let $m \geq 1$ be an integer.*
(i) *If $m < t + t/(p-1)$, we have*

$$\mathrm{gr}^m(\phi) \simeq \begin{cases} k, & \text{if } p \nmid m, \\ 0, & \text{if } p \mid m. \end{cases}$$

(ii) *If $m = t + t/(p-1)$, we have $\mathrm{gr}^{t+t/(p-1)}(\phi) \simeq k/(a + \overline{a}_p C^{-1})k$, where a is the residue class of $a_1 \pi^{-t}$. If we further assume that k is separably closed, then $\mathrm{gr}^{t+t/(p-1)}(\phi) = 0$. If k is quasi-finite, then $\mathrm{gr}^{t+t/(p-1)}(\phi) \simeq \mathbb{Z}/p\mathbb{Z}$.*
(iii) *If $m > t + t/(p-1)$, we have $G^m(\phi) = 0$. In particular, $\mathrm{gr}^m(\phi) = 0$.*

Proof. The proof below is cited from [9]. The assertions follow from Lemmas 2.2 and 2.4. If k is quasi-finite, then the homomorphism $\phi : \mathrm{gr}^{t/(p-1)}(F) \to \mathrm{gr}^{t+t/(p-1)}(G)$ is extended to $\phi : \overline{k} \to \overline{k}$. Since $H^1(k, \overline{k}) = 1$ and $\mathrm{Ker}(\phi) \simeq \mathbb{Z}/p\mathbb{Z}$ as G_k-modules, we have $k/\phi(k) \simeq H^1(k, \mathrm{Ker}(\phi)) \simeq \mathbb{Z}/p\mathbb{Z}$. $\square$

Let $\phi : F \to G$ be an isogeny with finite height $n > 1$ and *assume* $F[\phi]$ is cyclic and $F[\phi] \subset F(K)$. Let $x_0 \in F(K)$ be the generator of the cyclic group $F[\phi]$. The subgroup $pF[\phi] \subset F[\phi]$ generated by $[p]x_0$ has order p^{n-1}, where $[p]$ is the multiplication by p map on F. From the theorem of Lubin ([6], Chap. IV, Thm. 4), there exists a formal group $G_1 := F/pF[\phi]$ defined over $\mathcal{O}_K$ and the isogeny ϕ factors as $\phi = \phi_1 \circ \psi$, where $\psi : F \to G_1$ is an isogeny over $\mathcal{O}_K$ such that $F[\psi] = pF[\phi]$ (thus ψ is an isogeny with height

$n-1$ and ϕ_1 has height 1). Note that the kernel $G_1[\phi_1]$ is generated by $\psi(x_0)$. From the following lemma, the structure of $\mathrm{gr}^m(\phi)$ is obtained from that in the case of height 1 (Cor. 2.5).

Lemma 2.6. *Put* $t_1 := v_K(D(\phi_1))$.
(i) *If* $m < t_1 + t_1/(p-1)$, *the sequence*

$$0 \to \mathrm{gr}^{m/p}(\psi) \xrightarrow{\phi_1} \mathrm{gr}^m(\phi) \to \mathrm{gr}^m(\phi_1) \to 0$$

is exact, where $\mathrm{gr}^x(\psi) = 0$ *if* $x \notin \mathbb{Z}$ *by convention.*
(ii) *If* $m = t_1 + t_1/(p-1)$, *the sequence*

$$\mathrm{gr}^{t_1/(p-1)}(\psi) \xrightarrow{\phi_1} \mathrm{gr}^{t_1+t_1/(p-1)}(\phi) \to \mathrm{gr}^{t_1+t_1/(p-1)}\,\mathrm{gr}(\phi_1) \to 0$$

is exact.
(iii) *If* $m > t_1 + t_1/(p-1)$, *then the sequence*

$$0 \to \mathrm{gr}^{m-t_1}(\psi) \xrightarrow{\phi_1} \mathrm{gr}^m(\phi) \to \mathrm{gr}^m(\phi_1) \to 0$$

is exact. In particular, we have an isomorphism $\mathrm{gr}^{m-t_1}(\psi) \simeq \mathrm{gr}^m(\phi)$.

Proof. (i) and (ii); $m \le t_1 + t_1/(p-1)$. In the commutative diagram

$$\begin{tikzcd}
\mathrm{gr}^{m/p}(G_1) \arrow[r, "\phi_1"] \arrow[d, two heads] & \mathrm{gr}^m(G) \arrow[r] \arrow[d, two heads] & \mathrm{gr}^m(\phi_1) \arrow[r] \arrow[d, equal] & 0 \\
\mathrm{gr}^{m/p}(\psi) \arrow[r, "\phi_1"] & \mathrm{gr}^m(\phi) \arrow[r] & \mathrm{gr}^m(\phi_1) \arrow[r] & 0
\end{tikzcd} \qquad (2)$$

the top horizontal row is exact by Lemma 2.4 and the vertical arrows are surjective. We show the injectivity of $\phi_1 : \mathrm{gr}^{m/p}(\psi) \to \mathrm{gr}^m(\phi)$ when $m < t_1 + t_1/(p-1)$ and $m \mid p$. In this case, the map $\phi_1 : \mathrm{gr}^{m/p}(G_1) \to \mathrm{gr}^m(G)$ in (2) is injective. Thus, it is enough to show the surjectivity of $\phi_1 : G_1^{m/p}(K) \cap \psi F(K)/G_1^{m/p+1}(K) \cap \psi F(K) \to G^m(K) \cap \phi F(K)/G^{m+1}(K) \cap \phi F(K)$. For any $\phi(x) = \phi_1 \circ \psi(x)$ in $G^m(K) \cap \phi F(K)/G^{m+1}(K)$, there exists $y \in G_1^{m/p}(K)$ such that $\phi_1(y) = \phi(x)$ by Lemma 2.4. Hence, we obtain $y = \psi(x) \in G_1^{m/p}(K) \cap \psi F(K)/G_1^{m/p+1}(K) \cap \psi F(K)$. The assertion in (iii) follows from the similar argument as above. $\square$

Inductively, one can find isogenies $\phi_i : G_i \to G_{i-1}$ with height 1 such that $\phi = \phi_1 \circ \cdots \circ \phi_n$ and $G_i = F/p^i F[\phi]$, where $p^i F[\phi]$ is the subgroup of $F[\phi]$ generated by $[p^i]x_0$ (we denoted by $F = G_n$ and $G = G_0$ by convention). Define $t_i := v_K(D(\phi_i))$ and put $t_0 := 0$.

Lemma 2.7. *For $1 \leq i < n$, we have $t_i \leq t_{i+1}$ and $p^{n-i} \mid t_i$. The equality $t_i = t_{i+1}$ does not hold if the height of $F > 1$.*

Proof. By induction on n, it is enough to show the case $n = 2$; $\phi = \phi_1 \circ \phi_2$ has height 2. Recall $[p]x_0 \in F[\phi_2]$ and $\phi_2(x_0) \in G_1[\phi_1]$. From Lemma 2.1, we obtain $t_1/(p-1) = v_K(\phi_2(x_0))$ and

$$t_2/(p-1) = v_K([p]x_0) = v_K(\widehat{\phi}_2 \circ \phi_2(x_0)) \geq v_K(\phi_2(x_0)). \tag{3}$$

Hence $t_2 \geq t_1$ and $v_K(\phi_2(x_0)) = p v_K(x_0)$. From the inequality (3), if the height of $F > 1$, then we have $t_i < t_{i+1}$. $\square$

Proposition 2.8. *Let $\phi = \phi_1 \circ \cdots \circ \phi_n : F \to G$ and t_i be as above. Put $c_i(\phi) := t_0 + t_1 + \cdots + t_i + t_i/(p-1)$ for $0 \leq i \leq n$.*
(i) *If $c_i(\phi) < m < c_{i+1}(\phi)$ for some $0 \leq i < n$, then we have*

$$\mathrm{gr}^m(\phi) \simeq \mathrm{gr}^{m-(t_1+t_2+\cdots+t_i)}(\phi_{i+1} \circ \cdots \circ \phi_n) \simeq \begin{cases} k, & \text{if } p^{n-i} \nmid m, \\ 0, & \text{if } p^{n-i} \mid m. \end{cases}$$

(ii) *If $m = c_{i+1}(\phi)$, for some $0 \leq i < n$, we have*

$$\mathrm{gr}^{c_{i+1}(\phi)}(\phi) \simeq \mathrm{gr}^{pt_{i+1}/(p-1)}(\phi_{i+1} \circ \cdots \circ \phi_n) \simeq k/(a + \overline{a}_p C^{-1})k,$$

where a_p is the coefficient of T^p in $\phi_{i+1} \in \mathcal{O}_K[\![T]\!]$ and a is the residue class of $D(\phi_{i+1})\pi^{-t_{i+1}}$. If we further assume that k is separably closed, then $\mathrm{gr}^{c_{i+1}(\phi)}(\phi) = 0$. If k is quasi-finite, then $\mathrm{gr}^{c_{i+1}(\phi)}(\phi) \simeq \mathbb{Z}/p\mathbb{Z}$.
(iii) *If $m > c_n(\phi)$ then we have $G^m(\phi) = 0$. In particular $\mathrm{gr}^m(\phi) = 0$.*

Proof. From Lemma 2.5, we may assume $n > 1$. Put $\psi = \phi_2 \circ \cdots \circ \phi_n$. First we consider the case $0 < m < c_1(\phi)$ in (i). In the exact sequence (Lem. 2.6 (i))

$$0 \to \mathrm{gr}^{m/p}(\psi) \to \mathrm{gr}^m(\phi) \to \mathrm{gr}^m(\phi_1) \to 0,$$

the isogeny ϕ_1 has height 1 and ψ has height $n-1$. Thus we obtain the structure of $\mathrm{gr}^m(\phi)$ for $m < c_1(\phi)$ by induction on n and Lemma 2.5 (i). If $m = c_1(\phi)$,

$$\mathrm{gr}^{t_1/(p-1)}(\psi) \to \mathrm{gr}^{t_1+t_1/(p-1)}(\phi) \to \mathrm{gr}^{t_1+t_1/(p-1)}(\phi_1) \to 0$$

by Lemma 2.6 (ii). From Lemma 2.7, we have $p^{n-1} \mid t_1$. By (i) and the case $m < c_1(\phi)$, $\mathrm{gr}^{t_1+t_1/(p-1)}(\psi) = 0$ and thus $\mathrm{gr}^{t_1+t_1/(p-1)}(\phi) \simeq \mathrm{gr}^{t_1+t_1/(p-1)}(\phi_1)$. The assertion follows from Lemma 2.5 (ii). Consider the case $m > c_1(\phi)$ in (i) and (ii). By Lemma 2.6 (iii), $\mathrm{gr}^{m-t_1}(\psi) \simeq \mathrm{gr}^m(\phi)$. From the induction on n, the assertions are reduced to the case $m \leq c_1(\phi)$. $\square$

Let L/K be a finite Galois extension with Galois group $H = \mathrm{Gal}(L/K)$. Recall that we call x is a *jump* for the ramification filtration $(H_j)_{j \geq -1}$ in the lower numbering (resp. $(H^j)_{j \geq -1}$ in the upper numbering) of H if $H_x \neq H_{x+\varepsilon}$ (resp. $H^x \neq H^{x+\varepsilon}$) for all $\varepsilon > 0$ (for definition of the ramification subgroups, see [17], Chap. IV).

Proposition 2.9. *For $y \in G^m(\phi) \smallsetminus G^{m+1}(\phi)$, take $x \in F(\overline{K})$ with $\phi(x) = y$ in $G(\phi)$. If $1 \leq m < c_1(\phi)$ and $p \nmid m$, then the definition field $L = K(x)$ of x over K is totally ramified Galois extension of degree p^n. The jumps of $H := \mathrm{Gal}(L/K)$ in the upper numbering are $c_1(\phi) - m, \ldots, c_n(\phi) - m$. In particular, $H^{c_n(\phi)-m} \neq 1$.*

Proof. For $n = 1$; namely $\phi = \phi_1$ has height 1, the assertion follows from [9], Lemma 2.1.5. For $n > 1$, for the isogeny $\phi = \psi \circ \phi_1$ ($\psi = \phi_2 \circ \cdots \circ \phi_n$), we have $y' \in G_1(\overline{K})$ such that $\psi(x) = y'$ and $\phi_1(y') = y$. The isogeny ϕ_1 has height 1, the extension $K' := K(y')/K$ is totally ramified extension of degree p. The jump is $pt_1/(p-1) - m$. Since $m < c_1(\phi)$ and $p \nmid m$, $v_{K'}(y) = pv_K(y) = v_K(\phi_1(y')) = pv_{K'}(y')$. Hence $v_{K'}(y') = v_K(y) = m$. By induction on n, the extension L/K' is totally ramified extension of degree p^{n-1}. The jumps of $\mathrm{Gal}(L/K')$ in the lower numbering are $p^2t_2/(p-1) - m, p^3t_3/(p-1) - m, \ldots, p^nt_n/(p-1) - m$. Since the ramification subgroups in the lower numbering commutes with subgroups and in the upper numbering commutes with quotients ([17], Chap. IV), the jumps of the ramification subgroups of H in the lower numbering are $p^it_i/(p-1) - m$ for $1 \leq i \leq n$. The ramification subgroups H^s of H in the upper numbering is defined by the Herbrand function φ of H as $H_j = H^{\varphi(j)}$. For the positive integer m, we have $\varphi(m) + 1 = \sum_{i=0}^{m} \#(H_i/H_0)$. Thus $\varphi(p^it_i/(p-1) - m) = c_i(\phi) - m$. $\square$

The isogeny $[p^n] : \widehat{\mathbb{G}}_m \to \widehat{\mathbb{G}}_m$ defined by multiplication by p^n on the multiplicative group $\widehat{\mathbb{G}}_m$ has the kernel $\widehat{\mathbb{G}}_m[p^n] = \mu_{p^n}$ which is cyclic of order p^n. If K contains a p^n-th root of unity ζ_{p^n}, $\widehat{\mathbb{G}}_m[p^n] \subset \widehat{\mathbb{G}}_m(K)$. Note also the filtration $\widehat{\mathbb{G}}_m^j([p^n])$ of $\widehat{\mathbb{G}}_m([p^n]) = \widehat{\mathbb{G}}_m(K)/[p^n]\widehat{\mathbb{G}}_m(K) \subset K^\times/p^n$ associated with $[p^n]$ is the image of the higher unit groups $U_K^j = 1 + \mathfrak{m}_K^j$ in $K^\times/p^n$

which is also dented by U_n^j, namely, $\widehat{\mathbb{G}}_m^j([p^n]) = U_n^j := U_K^j/((K^\times)^{p^n} \cap U_K^j)$. Put $U_n^0 := K^\times/p^n$ and let $\mathrm{gr}(p^n)$ be the graded group ($= \mathrm{gr}\, k_{1,n}$ in terms of the appendix) associated with the filtration $(U_n^m)_{m\geq 0}$;

$$\mathrm{gr}(p^n) := \bigoplus_{m\geq 0} \mathrm{gr}^m(p^n), \quad \mathrm{gr}^m(p^n) := U_n^m/U_n^{m+1}. \tag{4}$$

The isogeny $[p^n]$ factors as $[p^n] = [p] \circ \cdots \circ [p]$ (n times). In particular, $c_i := c_i([p^n]) = ie + e_0$, where $e := v_K(p)$ and $e_0 := e/(p-1)$. Let $\phi : F \to G$ be an isogeny with height n as in Proposition 2.8. Fix an isomorphism $F[\phi] \simeq \mu_{p^n} = \widehat{\mathbb{G}}_m[p^n]$. The isogeny induces the Kummer homomorphism $\delta : G(\phi) \to H^1(K, F[\phi]) = K^\times/p^n$. We compare the filtration $G^m(\phi)$ and the filtration $\widehat{\mathbb{G}}_m^j([p^n]) = U_n^j$ on $K^\times/p^n$. In the case of height 1 we have the following theorem:

Theorem 2.10 ([9], Thm. 2.1.6)**.** *Let $\phi : F \to G$ be an isogeny defined over $\mathcal{O}_K$ of height 1, and $t := v_K(D(\phi))$. Assume that $F[\phi] \subset F(K)$ and $\zeta_p \in K$. Then, the Kummer map δ induces a bijection $\delta : G^m(\phi) \xrightarrow{\simeq} U_1^{pe_0 - pt/(p-1)+m}$ for any $m \geq 1$.*

We extend the above theorem to the case of height > 1 assuming $\zeta_{p^n} \in K$. Let $\phi = \phi_1 \circ \cdots \circ \phi_n : F \to G$ and t_i be as in Proposition 2.8. Put $c_i(\phi) := t_0 + t_1 + \cdots + t_i + t_i/(p-1)$ for $0 \leq i \leq n$. First we show $\delta(G^m(\phi)) \subset U_n^{pe_0 - pt_1/(p-1)+m}$ for $m < c_1(\phi) = t_1 + t_1/(p-1)$ with $m \nmid p$. Let j be the biggest integer such that $\delta(G^m(\phi)) \subset U_n^j$. Because of $p \nmid m$, δ induces a non-zero homomorphism $\mathrm{gr}^m(\phi) \to \mathrm{gr}^j(p^n)$. From Lemma 2.6 (i), we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{gr}^m(\phi) \arrow[r, "\simeq"] \arrow[d, "\delta"'] & \mathrm{gr}^m(\phi_1) \arrow[d, "\delta"] \\
\mathrm{gr}^j(p^n) \arrow[r] & \mathrm{gr}^j(p)
\end{tikzcd}$$

where the top horizontal homomorphism is an isomorphism. Since the left δ is non-zero, Theorem 2.10 implies $j = pe_0 - pt_1/(p-1) + m$. In particular we obtain $\delta(G(\phi)) \subset U_n^{e+e_0-pt_1/(p-1)+1}$. Next, we show that δ induces a bijection $\mathrm{gr}^m(\phi) \xrightarrow{\simeq} \mathrm{gr}^{c_i - c_i(\phi)+m}(p^n)$ on the graded groups by induction on i, where $c_i = ie+e_0$. From Proposition 2.8 and the above observation, (although we do not discuss on m with $p \mid m$) the map δ induces $\mathrm{gr}^m(\phi) \simeq \mathrm{gr}^{pe_0 - pt_1/(p-1)+m}(p^n)$

for any $m < c_1(\phi)$. For $m = c_1(\phi)$, let j be the biggest integer such that $\delta(G^{t_1+t_1/(p-1)}(\phi)) \subset U_n^j$ as above. From Lemma 2.6 (b), we have

$$\begin{tikzcd}
\mathrm{gr}^{t_1+t_1/(p-1)}(\phi) \arrow[r, "\simeq"] \arrow[d, "\delta"'] & \mathrm{gr}^{t_1+t_1/(p-1)}(\phi_1) \arrow[d, "\delta"] \\
\mathrm{gr}^j(p^n) \arrow[r] & \mathrm{gr}^j(p)
\end{tikzcd}.$$

Thus $j = e + e_0 = e + e_0 - pt_1/(p-1) + m$. We obtain

$$\mathrm{gr}^m(\phi) \xrightarrow{\simeq} \mathrm{gr}^{e+e_0-pt_1/(p-1)+m}(p^n) \tag{5}$$

for $m \le c_1(\phi)$. For $c_i(\phi) < m \le c_{i+1}(\phi)$ with $i > 1$, let j be the biggest integer such that $\delta(G^m(\phi)) \subset U_n^j$ again. From the induction hypothesis, $j > c_i = ie + e_0$. By Proposition 2.8 we have the commutative diagram:

$$\begin{tikzcd}
\mathrm{gr}^{m-(t_1+t_2+\cdots+t_i)}(\phi_{i+1} \circ \cdots \circ \phi_n) \arrow[r, "\simeq"] \arrow[d, "\delta"'] & \mathrm{gr}^m(\phi) \arrow[d, "\delta"] \\
\mathrm{gr}^{j-ie}(p^{n-i}) \arrow[r, "\simeq"] & \mathrm{gr}^j(p^n)
\end{tikzcd}.$$

Hence $m - (t_1 + t_2 + \cdots + t_i) \le pt_{i+1}/(p-1)$. By the argument above (5) we obtain $j = c_{i+1} - c_{i+1}(\phi) + m$. From Proposition 2.8, we obtain the following theorem:

Theorem 2.11. *The image of the Kummer map $\delta : G(\phi) \to K^\times/p^n$ is contained in $U_n^{c_1-c_1(\phi)+1}$ and δ induces a bijection $\mathrm{gr}^m(\phi) \xrightarrow{\simeq} \mathrm{gr}^{c_i-c_i(\phi)+m}(p^n)$, for m with $c_{i-1}(\phi) < m \le c_i(\phi)$.*

3 Cycle map

Let K be a finite extension field over $\mathbb{Q}_p$. Let $X = E \times E'$ be the product of two elliptic curves E and E' over K with $E[p^n]$ and $E'[p^n]$ are K-rational. The goal of this section is to calculate the image of the Albanese kernel $T(X) := \mathrm{Ker}(A_0(X) \to X(K))$ by the cycle map $\rho : A_0(X) \to H^4(X, \mathbb{Z}/p^n(2))$. From the argument below which is essentially same as in the proof of Theorem 4.3 in [22], the study of the image of $T(X)/p^n$ boils down to the calculation of the image of the Kummer map $\delta : E(K) \to H^1(K, E[p^n])$ and the Hilbert

symbol: The image of $T(X)$ is contained in $H^2(K, E[p^n] \otimes E'[p^n])$ a direct summand of the étale cohomology group $H^4(X, \mathbb{Z}/p^n(2))$. The Albanese kernel $T(X)$ is isomorphic to the *Somekawa K-group* $K(K; E, E')$ defined by some quotient of $\bigoplus_{K'/K} E(K') \otimes E'(K')$, where K' runs through all finite extensions of K (for definition, see [19], [16]). Thus there is a natural surjection $\bigoplus_{K'/K} E(K') \otimes E'(K') \to T(X)/p^n$. The cycle map also induces the following commutative diagram (*cf.* Proof of Prop. 2.4 in [22]):

$$\begin{tikzcd}
\bigoplus_{K'/K} E(K') \otimes E'(K') \arrow[r, "\delta\otimes\delta'"] \arrow[dd] & \bigoplus_{K'/K} H^1(K', E[p^n]) \otimes H^1(K', E'[p^n]) \arrow[d, "\cup"] \\
 & \bigoplus_{K'/K} H^2(K', E[p^n] \otimes E'[p^n]) \arrow[d, "N"] \\
T(X)/p^n \arrow[r, "\rho"] & H^2(K, E[p^n] \otimes E'[p^n])
\end{tikzcd},$$

where δ (resp. δ') is the Kummer map $\delta : E(K') \to H^1(K', E[p^n])$ (resp. $\delta : E'(K') \to H^1(K', E'[p^n])$), $\cup$ is the cup product and N is the norm map. From the calculation below, the image of the cup product does not depend on a extension K'/K. So we consider the case $K' = K$ only. If we fix isomorphisms $E[p^n] \simeq \mu_{p^n} \oplus \mu_{p^n}$ and $E'[p^n] \simeq \mu_{p^n} \oplus \mu_{p^n}$, the cup product is characterized by the Hilbert symbol $(\ , \ )_n : K^\times \times K^\times \to \mu_{p^n}$ as follows (*cf.* [17], Chap. XIV, Prop. 5):

$$\begin{tikzcd}
H^1(K, E[p^n]) \otimes H^1(K, E'[p^n]) \arrow[r, "\cup"] \arrow[d, "\simeq"'] & H^2(K, E[p^n] \otimes E'[p^n]) \arrow[d, "\simeq"] \\
(K^\times/p^n)^{\oplus 2} \otimes (K^\times/p^n)^{\oplus 2} \arrow[r, "(\ , \ )_n^{\oplus 4}"] & (\mu_{p^n})^{\oplus 4}
\end{tikzcd}.$$

Recall that the Hilbert symbol is defined by $(a, b)_n := \rho_K(a)(\sqrt[p^n]{b})/(\sqrt[p^n]{b})$, for $a, b \in K^\times$, where $\rho_K : K^\times \to G_K^{\mathrm{ab}}$ is the reciprocity map. Recall that the filtration U_n^j on $K^\times/p^n$ is defined by the image of U_K^j in $K^\times/p^n$ for $j \geq 1$ and $U_n^0 := K^\times/p^n$. Their orders of the image in μ_{p^n} by the Hilbert symbol are calculated as follows:

Lemma 3.1 ([20], Prop. 2.8)**.** *Put* $c_i := ie + e_0$ *for* $1 \leq i \leq n$, $c_0 := 0$, *and* $c_{n+1} := \infty$.
(i) $\#(U_n^s, K^\times/p^n)_n = p^{n-i}$ *for* $c_i < s \leq c_{i+1}$.

(ii) *If* $p \nmid s$, *then* $\#(U_n^s, U_n^t)_n = p^{n-i}$ *for* $c_i < s+t \leq c_{i+1}$.
(iii) *If* $s, t > 0$, $p \mid s$ *and* $p \mid t$, *then* $\#(U_n^s, U_n^t)_n = p^{n-i}$ *for* $c_i \leq s+t < c_{i+1}$.

A proof of Lemma 3.1 is founded in [20]. It is proved by direct computation of the Herbrand function of the Kummer extension $K(\sqrt[p^n]{b})$ over K for some $b \in K^\times$. We present another proof using the study in the last section.

Proof of Lemma 3.1. (i) From Proposition 2.8 (or Thm. A.2), $s > c_n$ if and only if $U_n^s = 1$. Since the symbol $(\ ,\ )_n$ is non-degenerate, the condition $s > c_n$ is equivalent to $(U_n^s, K^\times/p^n)_n = 1$ for any n. It is known that $(a,b)_n^p = (a,b)_{n-1}$ for $a, b \in K^\times$ (*cf.* [5], Chap. IV, (5.1)). Because of $s > c_1$, the multiplication by p map induces $U_{n-1}^{s-e} \simeq U_n^s$ (Lem. 2.6 (c) or Lem. A.1). By induction on n and i, $(U_n^s, K^\times/p^n)_n \subset \mu_{p^{n-i}}$ if and only if $s > c_i$ for any n and $1 \leq i \leq n$.
(ii) As in the proof of (i), it is enough to show that, for any n, $(U_n^s, U_n^t)_n = 1$ if and only if $s+t > c_n$. For $a, b \in \mathcal{O}_K$, we have

$$(6)\quad \begin{aligned} (1+a\pi^s, 1+b\pi^t)_n &= (1+a\pi^s(1+b\pi^t), -a\pi^s)_n(1+ab\pi^{s+t}, 1+b\pi^t)_n^{-1} \\ &= (1+\frac{ab\pi^{s+t}}{1+a\pi^s}, -a\pi^s)_n^{-1}(1+ab\pi^{s+t}, 1+b\pi^t)_n^{-1}. \end{aligned}$$

Thus $(U_n^s, U_n^t)_n \subset (U_n^{s+t}, K^\times/p^n)_n$ (*cf.* [2], Lem. 4.1). If we assume $s+t > c_n$, then $(U_n^s, U_n^t)_n \subset (U_n^{s+t}, K^\times/p^n)_n = 1$ by (i). Conversely, we show $(U_n^s, U_n^t)_n \neq 1$ for $s+t \leq c_n$. We may assume $s \geq t$ and $s+t = c_n$ and hence $p \nmid t$. For $n = 1, 2$, Proposition 2.9 says $\mathrm{Gal}(K(\sqrt[p^n]{b})/K)^s \neq 1$. Since the reciprocity homomorphism $\rho_K : K^\times \to \mathrm{Gal}(K(\sqrt[p^n]{b})/K)$ maps the higher unit group U_K^s onto the ramification subgroup $\mathrm{Gal}(K(\sqrt[p^n]{b})/K)^s$ ([17], Chap. XV, Cor. 3), we obtain $(U_n^s, U_n^t)_n \neq 1$. For $n > 2$, we have $s > c_1$. Therefore, $(U_{n-1}^{s-e})^p = U_n^s$ (Lem. 2.6 (c) or Lem. A.1). By induction on $n > 2$, there exist $a \in U_{n-1}^{s-e}$ and $b \in U_{n-1}^t$ such that $(a,b)_{n-1} \neq 1$. The assertion follows from $(a^p, b)_n = (a,b)_n^p = (a,b)_{n-1} \neq 1$.
(iii) As in the proof of (ii), it is enough to show that $(U_n^s, U_n^t)_n = 1$ if and only if $s+t \geq c_n$. If $s+t < c_n$, then $(U_n^s, U_n^t)_n \subset (U_n^{s+1}, U_n^t)_n \neq 1$ from (ii). Suppose $s+t \geq c_n$. By $U_n^0 = U_n^1$ and (ii) we may assume $s, t \leq 1$ and $s+t = c_n$. From (6), for $1+a\pi^s \in U_K^s, 1+b\pi^t \in U_K^t$, we have $(1+a\pi^s, 1+b\pi^t)_n^{-1} = (1+ab\pi^{s+t}/(1+a\pi^s), -a\pi^s)_n(1+ab\pi^{s+t}, 1+b\pi^t)_n$. From (ii) and $p \mid s$, we obtain $(1+a\pi^s, 1+b\pi^t)_n = 1$. $\square$

From the above lemma, the Hilbert symbol induces a homomorphism of graded groups: Let $M^0 := \mu_{p^n}$ and $M^m := \mu_{p^{n-i}}$ for $m > 0$ such that $c_i < m \leq c_{i+1}$. This filtration $(M^m)_{m\geq 0}$ makes μ_{p^n} a filtered group. The associated graded group $\mathrm{gr}(\mu_{p^n})$ is defined by $\mathrm{gr}(\mu_{p^n}) := \bigoplus_{m\geq 0}\mathrm{gr}^m(\mu_{p^n})$, where $\mathrm{gr}^m(\mu_{p^n}) := M^m/M^{m+1}$. On the other hand, let $\mathrm{gr}(p^n)$ be the graded group associated with the filtration $(U_n^m)_{m\geq 0}$ defined in (4). If $s, t > 0$, $p \mid s$ and $p \mid t$, the Hilbert symbol gives $(U_n^s, U_n^t)_n = M^{s+t+1}$. Otherwise $(U_n^s, U_n^t)_n = M^{s+t}$ (Lem. 3.1). We modify the structure of the graded tensor product $\mathrm{gr}(p^n)\otimes\mathrm{gr}(p^n)$ of the grade groups as follows: $\mathrm{gr}(p^n\otimes p^n) := \bigoplus_{m\geq 0}\mathrm{gr}^m(p^n\otimes p^n)$, where

$$\mathrm{gr}^m(p^n\otimes p^n) := \bigoplus_{\substack{m=s+t,\\ p\nmid s \text{ or } p\nmid t}} \mathrm{gr}^s(p^n)\otimes\mathrm{gr}^t(p^n) \oplus \bigoplus_{\substack{m=s+t+1,\\ p\mid s \text{ and } p\mid t}} \mathrm{gr}^s(p^n)\otimes\mathrm{gr}^t(p^n),$$

The symbol $(\ ,\ )_n : K^\times/p^n \otimes K^\times/p^n \to \mu_{p^n}$ induces $\mathrm{gr}(\ ,\ )_n : \mathrm{gr}(p^n\otimes p^n) \to \mathrm{gr}(\mu_{p^n})$. For any subgroups U and U' of $K^\times/p^n$, the induced graded subgroups $\mathrm{gr}(U)\subset\mathrm{gr}(p^n)$ and $\mathrm{gr}(U')\subset\mathrm{gr}(p^n)$ give the graded subgroup $\mathrm{gr}(U\otimes U)\subset\mathrm{gr}(p^n\otimes p^n)$. The order of the image $(U,U)_n$ coincides with that of the image of $\mathrm{gr}(U\otimes U)$ by $\mathrm{gr}(\ ,\ )_n$. Since the graded quotient $\mathrm{gr}^m(\mu_{p^n})$ is isomorphic to $\mathbb{Z}/p$ if $m = c_i$ for i and $\mathrm{gr}^m(\mu_{p^n}) = 0$ otherwise, this order is

$$\#(U,U')_n = p^\alpha, \quad \alpha := \#\{i \mid \mathrm{gr}^{c_i}(U\otimes U') \neq 1 \text{ for } 0 < i \leq n\}. \tag{7}$$

Next, we study the image of the map $\delta : E(K) \to H^1(K, E[p^n]) = K^\times/p^n \oplus K^\times/p^n$. When E has *split multiplicative reduction*, the uniformization theorem gives $K^\times/q^{\mathbb{Z}} \simeq E(K)$ for some $q \in K$.

Theorem 3.2 ([22], Lem. 4.5)**.** *Let E and F be elliptic curves over K which have split multiplicative reduction. Let $\phi : E \to F$ be an isogeny over K of degree p^n with cyclic kernel $E[\phi]$. Assume that the kernel $E[\phi]$ of ϕ and the kernel $F[\widehat{\phi}]$ of the dual isogeny $\widehat{\phi} : F \to E$ are K-rational. Then, the image of the Kummer map $\delta_\phi : E(K) \to H^1(K, E[\phi]) = K^\times/p^n$ is*

$$\mathrm{Im}(\delta_\phi) = \begin{cases} K^\times/p^n, & \text{if } \sqrt[p^n]{q} \notin E[\phi], \\ 1, & \text{if } \sqrt[p^n]{q} \in E[\phi]. \end{cases}$$

We choose an isomorphism $E[p^n] \simeq \mu_{p^n}\oplus\mu_{p^n}$ which maps $E[p^n] \supset \mathbb{G}_m[p^n] = \mu_{p^n}$ onto the second factor of $\mu_{p^n}\oplus\mu_{p^n}$. From the above theorem, we have

$$\mathrm{Im}(\delta) = K^\times/p^n \oplus 1 \tag{8}$$

when E has split multiplicative reduction.

We assume that E has *ordinary good reduction.* Let $\mathcal{E}$ be the Néron model of E over $\mathcal{O}_K$, $\widetilde{E}$ the neutral component of the special fiber of $\mathcal{E}$, and $\pi : E(K) = \mathcal{E}(\mathcal{O}_K) \to \widetilde{E}(k)$ the specialization map. The group $E(K) = \mathcal{E}(\mathcal{O}_K)$ has a filtration $E^i(K)$ $(i \geq 0)$ defined by $E^0(K) := \mathcal{E}(\mathcal{O}_K)$, $E^1(K) := \mathrm{Ker}(\pi)$ and for $i \geq 1$, $E^i(K) := \{(x,y) \in E(K) \mid v_K(x) \leq -2i\} \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is the origin on E. This filtration coincides with $\widehat{E}^i(K)$ of the formal group $\widehat{E}(K)$ defined in the previous section. Choose an isomorphism $E[p^n] \simeq \mu_{p^n} \oplus \mu_{p^n}$ which maps $E^1[p^n]$ onto the first factor of $\mu_{p^n} \oplus \mu_{p^n}$. Let x_0 be a generator of $E[p^n]$ and Φ be the subgroup of $E[p^n]$ generated by x_0. If $x_0 \in E^1[p^n]$, the isogeny $\phi : E \to F := E/\Phi$ has the cyclic kernel $E[\phi] = E^1[p^n]$. Since $E^1(K)$ is isomorphic to the formal group $\widehat{E}(K)$ and the height of $\widehat{E}$ (= the height of $[p]$) is 1, the isogeny $\phi : E \to F$ induces $[p^n] : \widehat{E} \to \widehat{E} \simeq \widehat{F}$. The first factor of the image of $\delta : E(K) \to H^1(K, E[p^n]) = K^\times/p^n \oplus K^\times/p^n$ coincides with the image of the Kummer map $\delta^1 : \widehat{E}(K) \to H^1(K, \widehat{E}[p^n]) = K^\times/p^n$. By Theorem 2.11, the image is U_n^1. On the other hand, if $x_0 \notin E^1[p^n]$, the isogeny $\phi : E \to F := E/\Phi$ has the kernel $E[\phi] \simeq \widetilde{E}[p^n]$. Hence, the image of $\delta_\phi : E(K) \to H^1(K, E[\phi])$ is contained in $H^1_{\mathrm{ur}}(K, E[\phi]) := \mathrm{Ker}(\mathrm{Res} : H^1(K, E[\phi]) \to H^1(K^{\mathrm{ur}}, E[\phi]))$, where K^{ur} is the completion of the maximal unramified extension of K and Res is the restriction map. The image of δ is contained in $U_n^1 \oplus H^1_{\mathrm{ur}}(K, \mu_{p^n})$. Mattuck's theorem [11] says $\#E(K)/p^n = ([K : \mathbb{Q}_p] + 2)p^n$. The order of $H^1_{\mathrm{ur}}(K, \mu_{p^n}) \simeq H^1(k, \mathbb{Z}/p^n)$ is p^n and $\#U_n^1 = ([K : \mathbb{Q}_p] + 1)p^n$. Thus

$$\mathrm{Im}(\delta) = U_n^1 \oplus H^1_{\mathrm{ur}}(K, \mu_{p^n}). \tag{9}$$

For the second factor, the restriction map $\mathrm{Res} : H^1(K, \mu_{p^n}) \to H^1(K^{\mathrm{ur}}, \mu_{p^n})$ induces $\mathrm{Res}^j : U_n^j/U_n^{j+1} \to U_n^{\mathrm{ur},j}/U_n^{\mathrm{ur},j}$, where $U_n^{\mathrm{ur},j}$ is the image of $U_{K^{\mathrm{ur}}}^j$ in $(K^{\mathrm{ur}})^\times/p^n$. Proposition 2.8 implies that Res^j is bijective if $j \neq ie + e_0$ for some $i \leq n$ and $\mathrm{Ker}(\mathrm{Res}^{c_i}) = U_n^{c_i}/U_n^{c_i+1} = \mathrm{gr}^{c_i}(p^n)$.

Finally, we consider that E has a *supersingular good reduction.* Let Φ be a subgroup generated by a generator of $E[p^n]$ and we denote by $\phi : E \to F := E/\Phi$ the induced isogeny. The first factor of the image of $\delta : E(K) \to H^1(K, E[p^n]) = K^\times/p^n \oplus K^\times/p^n$ is the image of the Kummer map $\delta_\phi : F(K) \to H^1(K, E[\phi]) = K^\times/p^n$ and another one is the image of the Kummer map $\delta_{\widehat{\phi}}$ associated with the dual isogeny $\widehat{\phi}$. Since the elliptic curve E has supersingular reduction, $F(K)/\phi E(K)$ is isomorphic to $F^1(K)/(\phi E(K) \cap F^1(K)) \simeq \widehat{F}(\phi)$ ([9], Lem. 3.2.3). As in the previous section, ϕ factors

as $\phi = \phi_1 \circ \cdots \circ \phi_n$ by height 1 isogenies ϕ_i. The invariants $t_i := D(\phi_i)$ satisfy $t_0 := 0 < t_1 < t_2 < \cdots < t_n < e$ (Lem. 2.7, see also Thm. 3.5). Theorem 2.11 says that the image of $\delta : E(K) \to K^\times/p^n$ is contained in $U_n^{e+e_0-(t_1+t_1/(p-1))+1}$. More precisely, one can describe the image in terms of the graded groups as follows: From Theorem 2.11, the graded quotient $\mathrm{gr}^m E := E^m(K)/E^{m+1}(K)$ maps onto $\mathrm{gr}^{c_i - c_i(\phi)+m}(p^n)$ for $c_{i-1}(\phi) < m \le c_i(\phi)$, where $c_i(\phi) := t_0 + t_1 + \cdots + t_i + t_i/(p-1)$ and $c_i := ie + e_0$. Hence δ induces a surjection

$$\mathrm{gr}(\delta) : \mathrm{gr}\, E := \bigoplus_{m\ge 0} \mathrm{gr}^m E \longrightarrow \bigoplus_{i=1}^{n} \bigoplus_{c_{i-1}(\phi)<m\le c_i(\phi)} \mathrm{gr}^{c_i-c_i(\phi)+m}(p^n).$$

Similarly, the dual isogeny $\widehat{\phi}$ is described by the dual isogenies $\widehat{\phi}_i$ of ϕ_i as $\widehat{\phi} = \widehat{\phi}_n \circ \cdots \circ \widehat{\phi}_1$. The invariants $\widehat{t}_i := D(\widehat{\phi}_{n-i+1}) = e - t_{n-i+1}$ satisfy $\widehat{t}_0 := 0 < \widehat{t}_1 < \widehat{t}_2 < \cdots < \widehat{t}_n < e$. Thus $c_i(\widehat{\phi}) = \widehat{t}_0 + \widehat{t}_1 + \cdots + \widehat{t}_i + \widehat{t}_i/(p-1)$. Summarize the above observations in terms of the graded groups, we have:

Theorem 3.3. *The Kummer map $\delta : E(K) \to H^1(K, E[p^n]) = K^\times/p^n \oplus K^\times/p^n$ induces $\mathrm{gr}(\delta) : \mathrm{gr}\, E \to \mathrm{gr}(p^n) \oplus \mathrm{gr}(p^n)$ on graded groups, where $\mathrm{gr}(p^n) := \bigoplus_{j\ge 0} \mathrm{gr}^j(p^n)$.*

(i) *If E has split multiplicative reduction,* $\mathrm{Im}(\mathrm{gr}(\delta)) = \mathrm{gr}(p^n) \oplus 1$.

(ii) *If E has ordinary reduction,*

$$\mathrm{Im}(\mathrm{gr}(\delta)) = \bigoplus_{j\ge 1} \mathrm{gr}^j(p^n) \oplus \bigoplus_{i=1}^{n} \mathrm{gr}^{c_i}(p^n).$$

(iii) *If E has supersingular reduction, then*

$$\mathrm{Im}(\mathrm{gr}(\delta)) = \bigoplus_{i=1}^{n} \bigoplus_{c_{i-1}(\phi)<m\le c_i(\phi)} \mathrm{gr}^{c_i-c_i(\phi)+m}(p^n) \oplus \bigoplus_{i=1}^{n} \bigoplus_{c_{i-1}(\widehat{\phi})<m\le c_i(\widehat{\phi})} \mathrm{gr}^{c_i-c_i(\widehat{\phi})+m}(p^n).$$

Now we complete the proof of the main theorem.

Theorem 3.4. *Let E and E' be elliptic curves over K with (semi-)stable reduction and $E[p^n]$ and $E'[p^n]$ are K-rational. The structure of the image of $T(X)/p^n$ for $X = E \times E'$ by the cycle map ρ is*

(i) $\mathbb{Z}/p^n$ *if both E and E' have ordinary or split multiplicative reduction.*

(ii) $\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$ *if E and E' have different reduction types.*

Proof. We denote the subsets of $\mathbb{N} := \mathbb{Z}_{\geq 0}$ which indicate the indexes of the graded quotients of $\mathrm{Im}(\mathrm{gr}(\delta))$ by $M := \{m \geq 0\}$, $O := \{m \geq 1\}$, $O_{\mathrm{ur}} := \{m = c_i \mid 0 < i \leq n\}$,

$$S := \bigcup_{i=1}^{n} \{c_i - \frac{pt_i - t_{i-1}}{p-1} < m \leq c_i\}, \text{ and}$$
$$\widehat{S} := \bigcup_{i=1}^{n} \{c_{i-1} + \frac{pt_{n-i+1} - t_{n-i+2}}{p-1} < m \leq c_i\},$$

where $t_{n+1} := e$ by convention. Define

$$d_j : E(K) \otimes E'(K) \overset{\delta\otimes\delta'}{\to} (K^\times/p^n \otimes K^\times/p^n)^{\oplus 4} \overset{\mathrm{pr}_j}{\to} K^\times/p^n \otimes K^\times/p^n,$$

where pr_j is the j-th projection. We calculate the order of the image of the composition $(\ ,\ )_n \circ d_j : E(K) \otimes E'(K) \to \mu_{p^n}$ for each j in the following five cases:

(a) Both of E and E' have split multiplicative reduction.
(b) Both of E and E' have ordinary reduction.
(c) E has ordinary reduction and E' has split multiplicative reduction.
(d) E has supersingular reduction and E' has split multiplicative reduction.
(e) E has supersingular reduction and E' has ordinary reduction.

First we consider the easiest case (a): Both of E and E' have split multiplicative reduction. From (8), the images of d_j are $K^\times/p^n \otimes K^\times/p^n$, $K^\times/p^n \otimes 1$, $1 \otimes K^\times/p^n$ and $1 \otimes 1$. By Lemma 3.1, the image of the cycle map is isomorphic to $\mathbb{Z}/p^n$.

Case (b): Both of E and E' have ordinary reduction. From (9), replace the index j if necessity, the image of d_j is $\mathrm{Im}(d_1) = U_n^1 \otimes U_n^1$, $\mathrm{Im}(d_2) = U_n^1 \otimes H^1_{\mathrm{ur}}(K, \mu_{p^n})$, $\mathrm{Im}(d_3) = H^1_{\mathrm{ur}}(K, \mu_{p^n}) \otimes U_n^1$ and $\mathrm{Im}(d_4) = H^1_{\mathrm{ur}}(K, \mu_{p^n}) \otimes H^1_{\mathrm{ur}}(K, \mu_{p^n})$. The image of $\mathrm{Im}(d_1)$ by the Hilbert symbol is μ_{p^n} (Lem. 3.1). We count the order of the image of d_2 in the graded groups. A subset R_i of $\mathbb{N} \times \mathbb{N}$ is define by

$$(10)\quad R_i := \{(s, ie+e_0-s) \mid 0 < s < ie+e_0,\ p \nmid s\} \cup \{(0, ie+e_0), (ie+e_0, 0)\}.$$

By (7), the order of $\mathrm{Im}(d_2)$ is p^α, where $\alpha = \#\{i \mid (O \times O_{\mathrm{ur}}) \cap R_i \neq \emptyset\}$. However, $(O \times O_{\mathrm{ur}}) \cap R_i = \emptyset$ for all i. Thus $\#\,\mathrm{Im}(d_2) = \#\,\mathrm{Im}(d_3) = 0$. Because $O_{\mathrm{ur}} \subset O$, we also obtain $\#\,\mathrm{Im}(d_4) = 0$.

Case (c): Assume that E has ordinary reduction and E' has split multiplicative reduction. Enough to consider the image of $U_n^1 \otimes K^\times/p^n$ and $H^1_{\mathrm{ur}}(K, \mu_{p^n}) \otimes K^\times/p^n$ by the Hilbert symbol. For the later, the required order is p^α, $\alpha = \#\{i \mid (O_{\mathrm{ur}} \times M) \cap R_i \neq \emptyset\}$. Since $O_{\mathrm{ur}} \times O \subset O_{\mathrm{ur}} \times M$, $\alpha = n$ from (b). By $O_{\mathrm{ur}} \times M \subset O \times M$, we obtain the order of the image of $U_n^1 \otimes K^\times/p^n$ is also p^n.

Case (d): E has supersingular reduction and E' has split multiplicative reduction. Since $O \times M \subset S \times M$ and $O_{\mathrm{ur}} \times M \subset \widehat{S} \times M$, the image is isomorphic to $\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$ by (c).

Case (e): E has supersingular reduction and E' has ordinary reduction. For each i, $(ie + e_0 - 1, 1) \in (S \times O) \cap R_i$ and $(ie + e_0 - 1, 1) \in (\widehat{S} \times O) \cap R_i$. On the other hand $S \times O_{\mathrm{ur}}, \widehat{S} \times O_{\mathrm{ur}} \subset O \times O_{\mathrm{ur}}$. Thus the image is isomorphic to $\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$. $\square$

When both of E and E' have supersingular reduction also, the computation of the image $\rho(T(X)/p^n)$ is done by the similar argument as in the proof of the above theorem. The results depend on the invariants $t_1 < t_2 < \cdots < t_n$ associated with the formal group $\widehat{E}$ and $t'_1 < t'_2 < \cdots < t'_n$ associated with $\widehat{E}'$ defined in the previous section. These invariants are calculated from the theory of the *canonical subgroup* due to Katz-Lubin. The canonical subgroup $H(E)$ of an elliptic curve E (when it exists) is a distinguished subgroup of order p in $\widehat{E}[p]$ which play the crucial role in the theory of overconvergent modular forms.

Theorem 3.5 ([8], Thm. 3.10.7; [3], Thm. 3.3)**.** *Let E be an elliptic curve over K with supersingular reduction. Let $a(\widehat{E})$ be the p-th coefficient of multiplication p formula $[p](T)$ of the formal group $\widehat{E}$.*

(i) *If $v_K(a(\widehat{E})) < pe/(p+1)$, then the canonical subgroup $H(\widehat{E}) \subset \widehat{E}[p]$ exists. For any non-zero $x \in \widehat{E}[p]$,*

$$v_K(x) = \begin{cases} \frac{e - v_K(a(\widehat{E}))}{p-1}, & \text{if } x \in H(\widehat{E}), \\ \frac{v_K(a(\widehat{E}))}{p^2 - p}, & \text{otherwise.} \end{cases}$$

For a subgroup $H \neq H(\widehat{E})$ of $\widehat{E}[p]$, $v_K(a(\widehat{E}/H)) = v_K(a(\widehat{E}))/p$ and the canonical subgroup $H(\widehat{E}/H)$ of the quotient $\widehat{E}/H$ is the canonical image of $\widehat{E}[p]$ in $\widehat{E}/H$. Moreover,

(a) *If* $v_K(a(\widehat{E})) < e/(p+1)$, *then* $v_K(a(\widehat{E}/H(\widehat{E}))) = pv_K(a(\widehat{E}))$. *The canonical image of* $\widehat{E}[p]$ *in* $\widehat{E}/H(\widehat{E})$ *is not the canonical subgroup of* $\widehat{E}/H(\widehat{E})$.

(b) *If* $v_K(a(\widehat{E})) = e/(p+1)$, *then* $v_K(a(\widehat{E}/H(\widehat{E}))) \geq pe/(p+1)$.

(c) *If* $e/(p+1) < v_K(a(\widehat{E})) < pe/(p+1)$, *then* $v_K(a(\widehat{E}/H(\widehat{E}))) = e - v_K(a(\widehat{E}))$ *and the canonical subgroup of* $\widehat{E}/H(\widehat{E})$ *is* $H(\widehat{E}/H(\widehat{E})) = \widehat{E}[p]/H$.

(ii) *If* $v_K(a(\widehat{E})) \geq pe/(p+1)$, *then* $v_K(x) = e/(p^2-1)$ *for any non-zero* $x \in \widehat{E}[p]$. *For any subgroup* H *of* $\widehat{E}[p]$, $v_K(a(\widehat{E}/H)) = e/(p+1)$ *and the canonical subgroup of the quotient* $\widehat{E}/H$ *is the image of* $\widehat{E}[p]$ *in* $\widehat{E}/H$.

For $n = 1$, let x_0 be a generator of $\widehat{E}[p]$ such that $v_K(x_0) = \max\{v_K(x) \mid 0 \neq x \in \widehat{E}[p]\}$. Let Φ be the subgroup of $\widehat{E}[p]$ generated by x_0. The induced isogeny $\phi : \widehat{E} \to \widehat{E}/\Phi$ has height 1. If $v_K(a(\widehat{E})) < pe/(p+1)$, $\Phi = \widehat{E}[\phi]$ is the canonical subgroup. Thus from Corollary 2.3 and Theorem 3.5, we have $t_1/(p-1) = v_K(x_0) = e_0 - v_K(a(\widehat{E}))/(p-1)$. If $v_K(a(\widehat{E})) \geq pe/(p+1)$, $t_1/(p-1) = v_K(x_0) = e_0/(p+1)$. Thus we obtain

Proposition 3.6. *The structure of the image of* $T(X)/p$ *for* $X = E \times E'$ *by the cycle map* ρ *is isomorphic to*

(i) $\mathbb{Z}/p \oplus \mathbb{Z}/p$ *if* $a(\widehat{E}) \neq a(\widehat{E}')$ *and* $a(\widehat{E}) + a(\widehat{E}') \neq e_0$,

(ii) $\mathbb{Z}/p$ *if* $a(\widehat{E}) = a(\widehat{E}') \neq e_0/2$, *or* $a(\widehat{E}) \neq a(\widehat{E}')$ *and* $a(\widehat{E}) + a(\widehat{E}') = e_0$,

(iii) 0 *if* $a(\widehat{E}) = a(\widehat{E}') = e_0/2$.

We conclude this note to give an example: Put $p = 5$ and suppose that E is an elliptic curve defined by $y^2 = x^3 + ax + b$ over K with $v_K(a) \geq 5e/6$ and $v_K(b) = 0$ (*cf.* [18], Sect. 1.11). Let us consider the self-product of the elliptic curve $X = E \times E$. Let Φ be the subgroup generated by a generator of $E[p^2]$. The induced isogeny $\phi : E \to F = E/\Phi$ factors as $\phi = \phi_1 \circ \phi_2$, where $\phi_i : F_i \to F_{i-1}$, $F_1 = E/pF[\phi]$ and $F_0 = F_2 = E$. By Theorem 3.5, we have $t_2/(p-1) = v_K(px_0) = e_0/(p+1)$, $v_K(a(\widehat{F_1})) = e/(p+1)$ and $t_1/(p-1) = v_K(\phi_2(x_0)) = e_0/p(p+1)$. Thus we have $S = (29e_0/6, 5e_0] \cup (41e_0/5, 9e_0]$, $\widehat{S} = (5e_0/6, 5e_0] \cup (5e_0, 9e_0]$, where $(s,t]$ is the subset of $\mathbb{N}$ consists of $n \in \mathbb{N}$ with $s < n \leq t$ and $p \nmid n$. It is easy to see $R_1 \cap (S \times S) = R_1 \cap (S \times \widehat{S}) = \emptyset$ and $R_1 \cap (\widehat{S} \times \widehat{S}) \neq \emptyset$. Here, the set R_i is defined in (10). If we *assume* $e_0 > 6$, then $R_2 \cap (S \times S) = \emptyset$. However, $R_2 \cap (S \times \widehat{S}), R_2 \cap (\widehat{S} \times \widehat{S})$ are non-empty. We obtain $\rho(T(X)/p^2) \simeq \mathbb{Z}/p^2 \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p$.

A Filtration on the Milnor K-groups

In higher dimensional local class field theory of Kato and Parshin, the Galois group of an abelian extension field on a q-dimensional local field K is described by the Milnor K-group $K_q^M(K)$ for $q \geq 1$. The information on the ramification is related to the natural filtration $U^m K_q$ which is by definition the subgroup generated by $\{1 + \mathfrak{m}_K^m, K^\times, \dots, K^\times\}$, where $\mathfrak{m}_K$ is the maximal ideal of the ring of integers $\mathcal{O}_K$. So it is important to know the structure of the graded quotients $\mathrm{gr}^m K_q := U^m K_q / U^{m+1} K_q$. In this appendix, we shall show that the results on the graded quotients in Section 2 associated with filtration on the multiplicative group (modulo p^n) work also on the Milnor K-groups. For a mixed characteristic Henselian discrete valuation field (abbreviated as hdvf in the following) which contains a p^n-th root of unity ζ_{p^n}, we determine the graded quotients $\mathrm{gr}^m k_{q,n}$ of the filtration of $k_{q,n} := K_q^M(K)/p^n K_q^M(K)$ instead of $\mathrm{gr}^m K_q$ in terms of differential forms of the residue field. J. Nakamura described $\mathrm{gr}^m k_{q,n}$ after determining $\mathrm{gr}^m K_q$ for all m when K is absolutely tamely ramified *i.e.,* the case of $(e, p) = 1$ ([13], Cor. 1.2). Although it is easy in the case of $q = 1$ (as in (1) in Sect. 2), the structure of $\mathrm{gr}^m K_q$ is still unknown in general. In particular, when K has mixed characteristic and (absolutely) wildly ramification, it is known only some special cases ([10], see also [14]). However, as in Section 2, to study $\mathrm{gr}^m k_{q,n}$ we use the structure of $\mathrm{gr}^m K_q$ only for lower m under the assumption $\zeta_{p^n} \in K$ (In [10], Kurihara treated a wildly ramified field with $\zeta_p \notin K$).

Let K be a hdvf of characteristic 0, and k its residue field of characteristic $p > 0$. Let $e = v_K(p)$ be the absolute ramification index of K and $e_0 := e/(p-1)$. For $m \geq 1$, let $U^m K_q$ be the subgroup of $K_q^M(K)$ defined as above. Put $U^0 K_q = K_q^M(K)$ and $\mathrm{gr}^m K_q := U^m K_q / U^{m+1} K_q$. Let $\Omega_k^1 := \Omega_{k/\mathbb{Z}}^1$ be the module of absolute Kähler differentials and Ω_k^q the q-th exterior power of Ω_k^1 over the residue field k. Define subgroups B_i^q and Z_i^q for $i \geq 0$ of Ω_k^q such that $0 = B_0^q \subset B_1^q \subset \cdots \subset Z_1^q \subset Z_0^q = \Omega_k^q$ by the relations $B_1^q := \mathrm{Im}(d : \Omega_k^{q-1} \to \Omega_k^q)$, $Z_1^q := \mathrm{Ker}(d : \Omega_k^q \to \Omega_k^{q+1})$, $C^{-1} : B_i^{q-1} \xrightarrow{\simeq} B_{i+1}^q / B_1^q$, and $C^{-1} : Z_i^q \xrightarrow{\simeq} Z_{i+1}^q / B_1^q$, where $C^{-1} : \Omega_k^q \xrightarrow{\simeq} Z_1^q / B_1^q$ is the inverse Cartier operator defined by

$$x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \mapsto x^p \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}. \tag{11}$$

First we recall the study on $\mathrm{gr}^m K_q$ for $m \leq e + e_0$ due to Bloch and

Kato which is an essential tool for our study. We fix a prime element π of K. For any m, we have a surjective homomorphism $\rho_m : \Omega_k^{q-1} \oplus \Omega_k^{q-2} \to \mathrm{gr}^m K_q$ defined by

$$\left(x\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}}, 0\right) \mapsto \{1 + \pi^m \widetilde{x}, \widetilde{y}_1, \ldots \widetilde{y}_{q-1}\},$$
$$\left(0, x\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-2}}{y_{q-2}}\right) \mapsto \{1 + \pi^m \widetilde{x}, \widetilde{y}_1, \ldots \widetilde{y}_{q-2}, \pi\},$$

where $\widetilde{x}$ and $\widetilde{y}_i$ are liftings of x and y_i. Note that the map ρ_m depends on a choice of π. Using this homomorphism, one can obtain the structure of the graded quotients $\mathrm{gr}^m K_q$ for any $m \le e + e_0$ ([2], see also [14]). Next we define the filtration $U^m k_{q,n}$ on $k_{q,n} = K_q^M(K)/p^n K_q^M(K)$, by the image of the filtration $U^m K_q$ on $k_{q,n}$. Our objective is to study the structure of its graded quotient $\mathrm{gr}^m k_{q,n} := U^m k_{q,n}/U^{m+1} k_{q,n}$. From the following lemma, we can investigate $\mathrm{gr}^m k_{q,n}$ for $m > e + e_0$ by its structure for $m \le e + e_0$.

Lemma A.1. *For $n > 1$ and $m > e + e_0$, the multiplication by p induces a surjective homomorphism $p : U^{m-e} k_{q,n-1} \to U^m k_{q,n}$. If we further assume $\zeta_{p^n} \in K$, then the map p is bijective.*

Proof. The surjectivity follows from the surjectivity of $p : U_K^{m-e} \to U_K^m$ (Lem. 2.2). To show the injectivity, for $x \in U^{m-e} K_q$ we assume that $px = p^n x'$ is in $p^n K_q^M(K) \cap U^m K_q$ for some $x' \in K_q^M(K)$. Thus $x - p^{n-1}x'$ is in the kernel of the multiplication by p on $K_q^M(K)$. It is known its kernel $= \{\zeta_p\} K_{q-1}^M(K)$. This fact was so called Tate's conjecture. It is a corollary of the Milnor-Bloch-Kato conjecture (due to Suslin, *cf.* [7], Sect. 2.4), now is a theorem of Voevodsky, Rost, and Weibel ([21]). Hence, for any i and $y \in K_{q-1}^M(K)$, we have $\{\zeta_p^i, y\} = p^{n-1}\{\zeta_{p^n}^i, y\}$. Thus we have $x \in p^{n-1} K_q^M(K)$. □

We determine $\mathrm{gr}^m k_{q,n}$ for any m and n as follows.

Theorem A.2. *We assume $\zeta_{p^n} \in K$. Let m and n be positive integers and s the integer such that $m = p^s m'$, $(m', p) = 1$. Put $c_i := ie + e_0$ for $i \ge 1$ and $c_0 := 0$.*

(i) *If $c_i < m < c_{i+1}$ for some $0 \le i < n$, we have*

$$\mathrm{gr}^m k_{q,n} \simeq \begin{cases} \mathrm{Coker}(\theta : \Omega_k^{q-2} \to \Omega_k^{q-1}/B_s^{q-1} \oplus \Omega_k^{q-2}/B_s^{q-2}), & \text{if } n - i > s, \\ \Omega_k^{q-1}/Z_{n-i}^{q-1} \oplus \Omega_k^{q-2}/Z_{n-i}^{q-2}, & \text{if } n - i \le s, \end{cases}$$

where θ is defined by $\omega \mapsto (C^{-s}d\omega, (-1)^q(m-ie)/p^sC^{-s}\omega)$.
(ii) *If $m = c_i$ for some $0 < i \leq n$,*

$$\mathrm{gr}^{ie+e_0}\, k_{q,n} \simeq (\Omega_k^{q-1}/(1+aC)Z_{n-i}^{q-1}) \oplus (\Omega_k^{q-2}/(1+aC)Z_{n-i}^{q-2}),$$

where C is the Cartier operator defined by

$$x^p\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}} \mapsto x\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}}.$$

(iii) *If $m > c_n$, then $U^m k_{q,n} = 0$.*

Note that the assertion of the case $m \leq e + e_0$ in the above theorem is due to Bloch-Kato ([2], Rem. 4.8).

Proof of Thm. A.2. As noted above, the assertion for $m \leq e + e_0 = c_1$ is known. It is known also $U^m k_{q,1} = 0$ for $m > e + e_0$ ([2], Lem. 5.1 (i)). So we assume $m > e + e_0$ and $n > 1$. Thus, for such m, we have an isomorphism $\mathrm{gr}^{m-e}\, k_{q,n-1} \xrightarrow{p} \mathrm{gr}^m\, k_{q,n}$ from the above lemma. By induction on n, we obtain the assertions. □

Corollary A.3. *If k is separably closed (we do not need the assumption $\zeta_{p^n} \in K$), then $\mathrm{gr}^{ie+e_0}\, k_{q,n} = 0$ for $i \geq 1$.*

Proof. The assertion follows from the fact $\mathrm{gr}^{e+e_0}\, k_{q,1} = 0$ ([2], Lem. 5.1 (ii)), Lemma A.1, and the induction on n. □

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